On a factorization of second-order elliptic operators and applications

2006 J. Phys. A: Math. Gen. 39 12407

(http://iopscience.iop.org/0305-4470/39/40/009)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **39** (2006) 12407–12425 doi:10.1088/0305-4470/39/40/009

On a factorization of second-order elliptic operators and applications

Vladislav V Kravchenko

Sección de Posgrado e Investigación Escuela Superior de Ingeniería Mecánica y Eléctrica
Instituto Politécnico Nacional, CP 07738 México, DF, Mexico

E-mail: vkravchenko@ipn.mx

Received 29 March 2006, in final form 31 July 2006
Published 19 September 2006
Online at stacks.iop.org/JPhysA/39/12407

Abstract

We show that given a nonvanishing particular solution of the equation

$$(\operatorname{div} p \operatorname{grad} + q)u = 0$$

the corresponding differential operator can be factorized into a product of two first-order operators. The factorization allows us to reduce the above equation to a first-order equation which in a two-dimensional case is a Vekua equation of a special form. Under quite general conditions on the coefficients p and q, we obtain an algorithm which allows us to construct in explicit form positive formal powers (solutions of the Vekua equation generalizing the usual powers $(z - z_0)^n$, $n = 0, 1, \ldots$). This result means that under quite general conditions one can construct an infinite system of exact solutions of the above equation explicitly and, moreover, at least when p and q are real valued this system will be complete in $\ker(\operatorname{div} p \operatorname{grad} + q)$ in the sense that any solution of the above equation in a simply connected domain Ω can be represented as an infinite series of obtained exact solutions which converges uniformly on any compact subset of Ω. Finally, we give a similar factorization of the operator $(\operatorname{div} p \operatorname{grad} + q)$ in a multidimensional case and obtain a natural generalization of the Vekua equation which is related to second-order operators in a similar way as its two-dimensional prototype does.

PACS numbers: 02.30.Fn, 02.30.Jr
Mathematics Subject Classification: 30G20, 30G35, 35J10, 35J15

1. Introduction

Consider the one-dimensional stationary Schrödinger equation

$$y'' + v(x)y = 0. \tag{1}$$

It is well known that given a nonvanishing particular solution y_0 of (1), the Schrödinger operator can be factorized as

$$\partial^2 + v(x) = \left(\partial + \frac{y_0'(x)}{y_0(x)}\right)\left(\partial - \frac{y_0'(x)}{y_0(x)}\right) \tag{2}$$

and as a consequence the general solution of (1) can be obtained.

In [18], it was shown that a similar factorization of a stationary Schrödinger operator is available in two dimensions. One of the results of the present work is the factorization of a more general two-dimensional elliptic operator $(\operatorname{div} p \operatorname{grad} + q)$ in a form similar to (2). The factorization allows us to reduce

$$(\operatorname{div} p \operatorname{grad} + q)u = 0 \tag{3}$$

to a Vekua equation of a special form. This Vekua equation becomes bicomplex if p or q are complex functions. One complex component of a solution of the Vekua equation (its real part when p and q are real valued) necessarily satisfies equation (3) and the other satisfies an associated equation having the form of (3) but with different coefficients p and q. This situation is a generalization of the fact that real and imaginary parts of an analytic function are harmonic, and likewise for any harmonic function in a simply connected domain its harmonic conjugate can be constructed, we obtain explicit formulae for constructing 'conjugate' solutions of the associated equations of the form (3). In the case $p \equiv 1$ and $q \equiv 0$, these formulae turn into the well known from complex analysis formulae for the construction of conjugate harmonic functions.

In [18], we established that under quite general conditions the positive formal powers corresponding to the Vekua equation (the pseudoanalytic functions generalizing the usual powers $(z - z_0)^n, n = 0, 1, \ldots$) can be constructed explicitly. Here, we develop this result and obtain that by a given nonvanishing particular solution of equation (3) under quite general conditions one can construct an infinite system of exact solutions of (3) explicitly and, moreover, at least when p and q are real valued this system will be complete in $\ker(\operatorname{div} p \operatorname{grad} + q)$ in the sense that any solution of (3) in a simply connected domain Ω can be represented as an infinite series of obtained exact solutions which converges uniformly on any compact subset of Ω.

In the final part of the present work, we obtain a factorization of the operator $(\operatorname{div} p \operatorname{grad} + q)$ in a multidimensional situation and reduce (3) to a first-order equation which generalizes the Vekua equation.

2. Preliminaries

A bicomplex number has the form

$$q = Q_1 + Q_2\mathrm{k}$$

where $Q_1 = q_0 + \mathrm{i}q_1$, $Q_2 = q_2 + \mathrm{i}q_3$, $q_j \in \mathbb{R}$, $j = \overline{0, 3}$; $\mathrm{i}^2 = \mathrm{k}^2 = -1$, $\mathrm{ik} = \mathrm{ki}$.

We will say that Q_1 and Q_2 are complex components of the bicomplex number q. Denote $\overline{q} = Q_1 - Q_2\mathrm{k}$. The corresponding conjugation operator we denote by C: $Cq = \overline{q}$. We will say that q is scalar if $Q_2 = 0$. That is in this case q is a usual complex number. Q_2 is called the vector part of Q. Sometimes we use the notation $Q_1 = \mathrm{Sc}(Q)$, $Q_2 = \mathrm{Vec}(Q)$.

The set of bicomplex zero divisors, that is of nonzero elements $q = Q_1 + Q_2\mathrm{k}$, $\{Q_1, Q_2\} \subset \mathbb{C}$ such that

$$q\overline{q} = (Q_1 + Q_2\mathrm{k})(Q_1 - Q_2\mathrm{k}) = 0, \tag{4}$$

we denote by $\mathfrak{S}$.

For further information on bicomplex numbers we refer to [24].

We will consider the variable $z = x + \mathrm{k}y$, where x and y are the real variables and the corresponding differential operators

$$\partial_{\overline{z}} = \tfrac{1}{2}(\partial_x + \mathrm{k}\partial_y) \qquad \text{and} \qquad \partial_z = \tfrac{1}{2}(\partial_x - \mathrm{k}\partial_y).$$

Notation $W_{\overline{z}}$ or W_z means the application of $\partial_{\overline{z}}$ or ∂_z respectively to a bicomplex function $W(z)$.

Note that if we consider

$$\partial_z \varphi = \Phi \tag{5}$$

in a whole complex plane or in a convex domain, where $\Phi = \Phi_1 + \mathrm{k}\Phi_2$ is a given bicomplex-valued function such that its scalar part Φ_1 and vector part Φ_2 satisfy

$$\partial_y \Phi_1 + \partial_x \Phi_2 = 0, \tag{6}$$

then there exists a scalar solution of (5) which can be reconstructed up to an arbitrary scalar constant c in the following way:

$$\varphi(x, y) = 2\left(\int_{x_0}^{x} \Phi_1(\eta, y)\,\mathrm{d}\eta - \int_{y_0}^{y} \Phi_2(x_0, \xi)\,\mathrm{d}\xi\right) + c \tag{7}$$

where (x_0, y_0) is an arbitrary fixed point in the domain of interest.

By A we denote the integral operator in (7):

$$A[\Phi](x, y) = 2\left(\int_{x_0}^{x} \Phi_1(\eta, y)\,\mathrm{d}\eta - \int_{y_0}^{y} \Phi_2(x_0, \xi)\,\mathrm{d}\xi\right) + c.$$

Note that formula (7) can be easily extended to any simply connected domain by considering the integral along an arbitrary rectifiable curve Γ leading from (x_0, y_0) to (x, y)

$$\varphi(x, y) = 2\left(\int_{\Gamma} \Phi_1\,\mathrm{d}x - \Phi_2\,\mathrm{d}y\right) + c.$$

Thus if Φ satisfies (6), there exists a family of scalar functions φ such that $\partial_z \varphi = \Phi$, given by the formula $\varphi = A[\Phi]$.

In a similar way, we define the operator $\overline{A}$ corresponding to $\partial_{\overline{z}}$:

$$\overline{A}[\Phi](x, y) = 2\left(\int_{x_0}^{x} \Phi_1(\eta, y)\,\mathrm{d}\eta + \int_{y_0}^{y} \Phi_2(x_0, \xi)\,\mathrm{d}\xi\right) + c.$$

3. Solutions of second-order elliptic equations as scalar parts of bicomplex pseudoanalytic functions

Consider

$$(-\Delta + \nu) f = 0 \tag{8}$$

in some domain $\Omega \subset \mathbf{R}^2$, where $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$, ν and f are the complex-valued (in our terms scalar) functions. We assume that f is a twice continuously differentiable function.

Theorem 1. *Let f be a nonvanishing in Ω particular solution of (8). Then for any complex-valued (scalar) function $\varphi \in C^2(\Omega)$ the following equalities hold:*

$$\frac{1}{4}(\Delta - \nu)\varphi = \left(\partial_{\overline{z}} + \frac{f_z}{f}C\right)\left(\partial_z - \frac{f_z}{f}C\right)\varphi = \left(\partial_z + \frac{f_{\overline{z}}}{f}C\right)\left(\partial_{\overline{z}} - \frac{f_{\overline{z}}}{f}C\right)\varphi. \tag{9}$$

Proof. Consider

$$\left(\partial_{\overline{z}} + \frac{f_z}{f}C\right)\left(\partial_z - \frac{f_z}{f}C\right)\varphi = \frac{1}{4}\Delta\varphi - \frac{|\partial_z f|^2}{f^2}\varphi - \partial_{\overline{z}}\left(\frac{\partial_z f}{f}\right)\varphi$$
$$= \frac{1}{4}\left(\Delta\varphi - \frac{\Delta f}{f}\varphi\right) = \frac{1}{4}(\Delta - \nu)\varphi. \qquad (10)$$

Thus, we have the first equality in (9). Now the application of C to both sides of (10) gives us the second equality in (9). □

In the case of a real-valued potential ν this theorem was proved in [18].

The following statement is known in a form of a substitution (see, e.g., [23]). Here, we formulate it as an operator relation.

Proposition 2. *Let p and q be complex-valued functions, $p \in C^2(\Omega)$ and $p \neq 0$ in Ω. Then*

$$\operatorname{div} p \operatorname{grad} + q = p^{1/2}(\Delta - r)p^{1/2} \qquad in \quad \Omega, \qquad (11)$$

where

$$r = \frac{\Delta p^{1/2}}{p^{1/2}} - \frac{q}{p}.$$

Proof. The easily verified relation

$$\operatorname{div} p \operatorname{grad} = p^{1/2}\left(\Delta - \frac{\Delta p^{1/2}}{p^{1/2}}\right)p^{1/2} \qquad (12)$$

is well known (see, e.g., [25]). Adding to both sides of (12) the term q (and representing it on the right-hand side as $p^{1/2}(q/p)p^{1/2}$) gives us (11). □

Theorem 3. *Let u_0 be a nonvanishing in Ω particular solution of*

$$(\operatorname{div} p \operatorname{grad} + q)u = 0 \qquad in \quad \Omega. \qquad (13)$$

Then under the conditions of proposition 2 for any complex-valued (scalar) continuously twice differentiable function φ the following equality holds:

$$\frac{1}{4}(\operatorname{div} p \operatorname{grad} + q)\varphi = p^{1/2}\left(\partial_z + \frac{f_{\overline{z}}}{f}C\right)\left(\partial_{\overline{z}} - \frac{f_{\overline{z}}}{f}C\right)p^{1/2}\varphi, \qquad (14)$$

where

$$f = p^{1/2}u_0. \qquad (15)$$

Proof. This is based on (9). From (11) we have that if u_0 is a solution of (13) then the function (15) is a solution of

$$(\Delta - r)f = 0. \qquad (16)$$

Then combining (11) and (9) we obtain (14). □

Remark 4. According to (12), $\Delta - r = f^{-1} \operatorname{div} f^2 \operatorname{grad} f^{-1}$ where f is a solution of (16). Then from (11) we have

$$\operatorname{div} p \operatorname{grad} + q = p^{1/2} f^{-1} \operatorname{div} f^2 \operatorname{grad} f^{-1} p^{1/2}. \qquad (17)$$

Taking (15) into account we obtain

$$\operatorname{div} p \operatorname{grad} + q = u_0^{-1} \operatorname{div} p u_0^2 \operatorname{grad} u_0^{-1} \qquad in \quad \Omega.$$

Remark 5. Let $q \equiv 0$. Then u_0 can be chosen as $u_0 \equiv 1$. Hence, (14) gives us the equality

$$\frac{1}{4}\operatorname{div}(p\operatorname{grad}\varphi) = p^{1/2}\left(\partial_z + \frac{\partial_{\bar z}p^{1/2}}{p^{1/2}}C\right)\left(\partial_{\bar z} - \frac{\partial_{\bar z}p^{1/2}}{p^{1/2}}C\right)(p^{1/2}\varphi).$$

In what follows we suppose that in Ω there exists a nonvanishing particular solution of (13) which we denote by u_0.

Let f be a scalar function of x and y. Consider the bicomplex Vekua equation

$$W_{\bar z} = \frac{f_{\bar z}}{f}\overline{W} \qquad in \quad \Omega. \tag{18}$$

Denote $W_1 = \operatorname{Sc} W$ and $W_2 = \operatorname{Vec} W$.

Remark 6. [18] Equation (18) can be written as follows:

$$f\partial_{\bar z}(f^{-1}W_1) + kf^{-1}\partial_{\bar z}(fW_2) = 0. \tag{19}$$

Theorem 7. *Let $W = W_1 + W_2\mathrm{k}$ be a solution of (18). Then $U = f^{-1}W_1$ is a solution of*

$$\operatorname{div}(f^2\nabla U) = 0 \qquad in \quad \Omega, \tag{20}$$

and $V = fW_2$ is a solution of

$$\operatorname{div}(f^{-2}\nabla V) = 0 \qquad in \quad \Omega, \tag{21}$$

the function W_1 is a solution of the stationary Schrödinger equation

$$-\Delta W_1 + r_1W_1 = 0 \qquad in \quad \Omega \tag{22}$$

with $r_1 = \Delta f/f$, and W_2 is a solution of the associated Schrödinger equation

$$-\Delta W_2 + r_2W_2 = 0 \qquad in \quad \Omega \tag{23}$$

where $r_2 = 2(\nabla f)^2/f^2 - r_1$ and $(\nabla f)^2 = f_x^2 + f_y^2$.

Proof. To prove the first part of the theorem we use the form of equation (18) given in remark 6. Multiplying (19) by f and applying ∂_z gives

$$\partial_z(f^2\partial_{\bar z}(f^{-1}W_1)) + \frac{k}{4}\Delta(fW_2) = 0$$

from where we have that $\operatorname{Sc}(\partial_z(f^2\partial_{\bar z}(f^{-1}W_1))) = 0$ which is equivalent to (20) where $U = f^{-1}W_1$.

Multiplying (19) by f^{-1} and applying ∂_z gives

$$\frac{1}{4}\Delta(f^{-1}W_1) + k\partial_z(f^{-2}\partial_{\bar z}(fW_2)) = 0$$

from where we have that $\operatorname{Sc}(\partial_z(f^{-2}\partial_{\bar z}(fW_2))) = 0$ which is equivalent to (21) where $V = fW_2$.

From (12) we have

$$(\Delta - r_1)W_1 = f^{-1}\operatorname{div}(f^2\nabla(f^{-1}W_1)).$$

Hence, from the just proven equation (20) we obtain that W_1 is a solution of (22).

In order to obtain equation (23) for W_2 it should be noted that

$$f\operatorname{div}(f^{-2}\nabla(fW_2)) = (\Delta - r_2)W_2. \qquad \square$$

In the case of a real-valued function f the relation between solutions of (18) and equations (20), (21) was observed in [20] and between solutions of (18) and equations (22), (23) in [18].

Remark 8. Observe that the pair of functions

$$F = f \qquad \text{and} \qquad G = \frac{k}{f} \tag{24}$$

is a generating pair for (18). This allows us to rewrite (18) in the form of an equation for pseudoanalytic functions of second kind:

$$\varphi_{\overline{z}} f + \psi_{\overline{z}} \frac{k}{f} = 0, \tag{25}$$

where φ and ψ are the scalar functions. If φ and ψ satisfy (25) then $W = \varphi f + \psi \frac{k}{f}$ is a solution of (18) and vice versa.

Denote $w = \varphi + \psi k$. Then from (25) we have

$$(w + \overline{w})_{\overline{z}} f + (w - \overline{w})_{\overline{z}} \frac{1}{f} = 0,$$

which is equivalent to

$$w_{\overline{z}} = \frac{1 - f^2}{1 + f^2} \overline{w}_{\overline{z}}. \tag{26}$$

The relation between (26) and (20), (21) in the case of a real-valued function f^2 was observed in [2] and resulted to be essential for solving the Calderón problem in the plane.

Theorem 9. *Let $W = W_1 + W_2 \mathrm{k}$ be a solution of (18). Assume that $f = p^{1/2} u_0$, where u_0 is a nonvanishing solution of (13) in Ω. Then $u = p^{-1/2} W_1$ is a solution of (13) in Ω and $v = p^{1/2} W_2$ is a solution of*

$$\left(\operatorname{div} \frac{1}{p} \operatorname{grad} + q_1\right) v = 0 \qquad in \quad \Omega, \tag{27}$$

where

$$q_1 = -\frac{1}{p}\left(\frac{q}{p} + 2\left\langle \frac{\nabla p}{p}, \frac{\nabla u_0}{u_0}\right\rangle + 2\left(\frac{\nabla u_0}{u_0}\right)^2\right). \tag{28}$$

Proof. According to theorem 7, the function $f^{-1} W_1$ is a solution of (20). From (17) we have that

$$p^{-1/2}(\operatorname{div} p \operatorname{grad} + q)(p^{-1/2} W_1) = f^{-1} \operatorname{div}(f^2 \nabla (f^{-1} W_1))$$

from which we obtain that $u = p^{-1/2} W_1$ is a solution of (13).

In order to obtain the second assertion of the theorem, let us show that

$$p^{1/2}\left(\operatorname{div} \frac{1}{p} \operatorname{grad} + q_1\right)(p^{1/2}\varphi) = f \operatorname{div}(f^{-2} \nabla (f\varphi))$$

for any scalar $\varphi \in C^2(\Omega)$. According to (12),

$$f \operatorname{div}(f^{-2} \nabla (f\varphi)) = \left(\Delta - \frac{\Delta f^{-1}}{f^{-1}}\right)\varphi = (\Delta - r_2)\,\varphi.$$

Straightforward calculation gives us the following equality:

$$\frac{\Delta f^{-1}}{f^{-1}} = \frac{3}{4}\left(\frac{\nabla p}{p}\right)^2 - \frac{1}{2}\frac{\Delta p}{p} + \left\langle \frac{\nabla p}{p}, \frac{\nabla u_0}{u_0}\right\rangle - \frac{\Delta u_0}{u_0} + 2\left(\frac{\nabla u_0}{u_0}\right)^2.$$

From the condition that u_0 is a solution of (13) we obtain the equality

$$-\frac{\Delta u_0}{u_0} = \frac{q}{p} + \left\langle \frac{\nabla p}{p}, \frac{\nabla u_0}{u_0} \right\rangle.$$

Thus,

$$\frac{\Delta f^{-1}}{f^{-1}} = \frac{3}{4}\left(\frac{\nabla p}{p}\right)^2 - \frac{1}{2}\frac{\Delta p}{p} + 2\left\langle \frac{\nabla p}{p}, \frac{\nabla u_0}{u_0} \right\rangle + \frac{q}{p} + 2\left(\frac{\nabla u_0}{u_0}\right)^2.$$

Note that

$$\frac{\Delta p^{-1/2}}{p^{-1/2}} = \frac{3}{4}\left(\frac{\nabla p}{p}\right)^2 - \frac{1}{2}\frac{\Delta p}{p}.$$

Then

$$\frac{\Delta f^{-1}}{f^{-1}} = \frac{\Delta p^{-1/2}}{p^{-1/2}} + 2\left\langle \frac{\nabla p}{p}, \frac{\nabla u_0}{u_0} \right\rangle + \frac{q}{p} + 2\left(\frac{\nabla u_0}{u_0}\right)^2.$$

Now taking q_1 in the form (28) we obtain the result from (11). □

Theorem 10 [18]. *Let W_1 be a solution of (22) in a simply connected domain Ω. Then the function W_2, solution of (23) such that $W = W_1 + W_2\mathrm{k}$ is a solution of (18) is constructed according to*

$$W_2 = f^{-1}\overline{A}(\mathrm{k}f^2\partial_{\bar{z}}(f^{-1}W_1)). \tag{29}$$

Given a solution W_2 of (23), the corresponding solution W_1 of (22) such that $W = W_1 + W_2\mathrm{k}$ is a solution of (18) is constructed as follows:

$$W_1 = -f\overline{A}(\mathrm{k}f^{-2}\partial_{\bar{z}}(fW_2)). \tag{30}$$

Remark 11. When in (22) $r_1 \equiv 0$ and $f \equiv 1$, equalities (29) and (30) turn into the well-known formulae in complex analysis for constructing conjugate harmonic functions.

Corollary 12. *Let U be a solution of (20). Then a solution V of (21) such that*

$$W = fU + \mathrm{k}f^{-1}V$$

is a solution of (18) is constructed according to

$$V = \overline{A}(\mathrm{k}f^2U_{\bar{z}}).$$

Conversely, given a solution V of (21), the corresponding solution U of (20) can be constructed as follows:

$$U = -\overline{A}(\mathrm{k}f^{-2}V_{\bar{z}}).$$

Proof. Consists in substitution of $W_1 = fU$ and of $W_2 = f^{-1}V$ into (29) and (30). □

Corollary 13. *Let $f = p^{1/2}u_0$, where u_0 is a nonvanishing solution of (13) in a simply connected domain Ω and u be a solution of (13). Then a solution v of (27) with q_1 defined by (28) such that $W = p^{1/2}u + \mathrm{k}p^{-1/2}v$ is a solution of (18) is constructed according to*

$$v = u_0^{-1}\overline{A}\big(\mathrm{k}pu_0^2\partial_{\bar{z}}\big(u_0^{-1}u\big)\big).$$

Let v be a solution of (27), then the corresponding solution u of (13) such that $W = p^{1/2}u + \mathrm{k}p^{-1/2}v$ is a solution of (18) is constructed according to

$$u = -u_0\overline{A}\big(\mathrm{k}p^{-1}u_0^{-2}\partial_{\bar{z}}(u_0v)\big).$$

Proof. Consists in substitution of $f = p^{1/2}u_0$, $W_1 = p^{1/2}u$ and $W_2 = p^{-1/2}v$ into (29) and (30). □

4. Some definitions and results from pseudoanalytic theory for bicomplex functions

4.1. Generating pair, derivative and antiderivative

Following [5] we introduce the notion of a bicomplex generating pair.

Definition 14. *A pair of bicomplex functions $F = F_1 + F_2\mathrm{k}$ and $G = G_1 + G_2\mathrm{k}$ possessing in Ω partial derivatives with respect to the real variables x and y is said to be a generating pair if it satisfies the inequality*

$$\mathrm{Vec}(\overline{F}G) \neq 0 \qquad in\ \Omega.$$

The following expressions are called characteristic coefficients of the pair (F, G)

$$a_{(F,G)} = -\frac{\overline{F}G_{\overline{z}} - F_{\overline{z}}\overline{G}}{F\overline{G} - \overline{F}G}, \qquad b_{(F,G)} = \frac{FG_{\overline{z}} - F_{\overline{z}}G}{F\overline{G} - \overline{F}G},$$

$$A_{(F,G)} = -\frac{\overline{F}G_{z} - F_{z}\overline{G}}{F\overline{G} - \overline{F}G}, \qquad B_{(F,G)} = \frac{FG_{z} - F_{z}G}{F\overline{G} - \overline{F}G}.$$

Every bicomplex function W defined in a subdomain of Ω admits the unique representation $W = \phi F + \psi G$ where the functions ϕ and ψ are scalar.

The (F, G)-derivative $\dot{W} = \frac{d_{(F,G)}W}{dz}$ of a function W exists and has the form:

$$\dot{W} = \phi_z F + \psi_z G = W_z - A_{(F,G)}W - B_{(F,G)}\overline{W} \tag{31}$$

if and only if

$$\phi_{\overline{z}} F + \psi_{\overline{z}} G = 0. \tag{32}$$

This last equation can be rewritten in the following form

$$W_{\overline{z}} = a_{(F,G)}W + b_{(F,G)}\overline{W} \tag{33}$$

which we call the bicomplex Vekua equation. Solutions of this equation are called (F, G)-pseudoanalytic functions.

Definition 15. *Let (F, G) and (F_1, G_1) be two generating pairs in Ω. (F_1, G_1) is called successor of (F, G) and (F, G) is called predecessor of (F_1, G_1) if*

$$a_{(F_1,G_1)} = a_{(F,G)} \qquad and \qquad b_{(F_1,G_1)} = -B_{(F,G)}.$$

The importance of this definition becomes obvious from the following statement.

Theorem 16. *Let W be an (F, G)-pseudoanalytic function and let (F_1, G_1) be a successor of (F, G). Then $\dot{W}$ is an (F_1, G_1)-pseudoanalytic function.*

Definition 17. *Let (F, G) be a generating pair. Its adjoint generating pair $(F, G)^* = (F^*, G^*)$ is defined by*

$$F^* = -\frac{2\overline{F}}{F\overline{G} - \overline{F}G}, \qquad G^* = \frac{2\overline{G}}{F\overline{G} - \overline{F}G}.$$

The (F, G)-integral is defined as follows:

$$\int_\Gamma W \,\mathrm{d}_{(F,G)}z = F(z_1)\mathrm{Sc}\int_\Gamma G^* W \,\mathrm{d}z + G(z_1)\mathrm{Sc}\int_\Gamma F^* W \,\mathrm{d}z$$

where Γ is a rectifiable curve leading from z_0 to z_1.

If $W = \phi F + \psi G$ is an (F, G)-pseudoanalytic function where ϕ and ψ are scalar functions then

$$\int_{z_0}^{z} \dot{W} \,\mathrm{d}_{(F,G)}z = W(z) - \phi(z_0)F(z) - \psi(z_0)G(z). \tag{34}$$

This integral is path independent and represents the (F, G)-antiderivative of $\dot{W}$.

4.2. Generating sequences and Taylor series in formal powers

Following [5] we introduce the following definitions and results.

Definition 18. *A sequence of generating pairs* $\{(F_m, G_m)\}$, $m = 0, \pm 1, \pm 2, \ldots$, *is called a generating sequence if* (F_{m+1}, G_{m+1}) *is a successor of* (F_m, G_m). *If* $(F_0, G_0) = (F, G)$, *we say that* (F, G) *is embedded in* $\{(F_m, G_m)\}$.

Theorem 19. *Let* (F, G) *be a generating pair in* Ω. *Let* Ω_1 *be a bounded domain,* $\overline{\Omega}_1 \subset \Omega$. *Then* (F, G) *can be embedded in a generating sequence in* Ω_1.

Definition 20. *A generating sequence* $\{(F_m, G_m)\}$ *is said to have period* $\mu > 0$ *if* $(F_{m+\mu}, G_{m+\mu})$ *is equivalent to* (F_m, G_m), *that is their characteristic coefficients coincide.*

Let W be an (F, G)-pseudoanalytic function. Using a generating sequence in which (F, G) is embedded we can define the higher derivatives of W by the recursion formula

$$W^{[0]} = W; \qquad W^{[m+1]} = \frac{d_{(F_m,G_m)} W^{[m]}}{\mathrm{d}z}, \qquad m = 1, 2, \ldots.$$

Definition 21. *The formal power* $Z_m^{(0)}(a, z_0; z)$ *with centre at* $z_0 \in \Omega$, *coefficient* a *and exponent 0 is defined as the linear combination of the generators* F_m, G_m *with complex constant coefficients* λ, μ *chosen so that* $\lambda F_m(z_0) + \mu G_m(z_0) = a$. *The formal powers with exponents* $n = 1, 2, \ldots$ *are defined by the recursion formula*

$$Z_m^{(n+1)}(a, z_0; z) = (n+1) \int_{z_0}^{z} Z_{m+1}^{(n)}(a, z_0; \zeta)\, \mathrm{d}_{(F_m,G_m)}\zeta. \tag{35}$$

This definition implies the following properties.

(1) $Z_m^{(n)}(a, z_0; z)$ is an (F_m, G_m)-pseudoanalytic function of z.
(2) If a' and a'' are scalar constants, then

$$Z_m^{(n)}(a' + ka'', z_0; z) = a' Z_m^{(n)}(1, z_0; z) + a'' Z_m^{(n)}(k, z_0; z).$$

(3) The formal powers satisfy the differential relations

$$\frac{\mathrm{d}_{(F_m,G_m)} Z_m^{(n)}(a, z_0; z)}{\mathrm{d}z} = n Z_{m+1}^{(n-1)}(a, z_0; z).$$

(4) The asymptotic formulae

$$Z_m^{(n)}(a, z_0; z) \sim a(z - z_0)^n, \qquad z \to z_0 \tag{36}$$

hold.

Assume now that

$$W(z) = \sum_{n=0}^{\infty} Z^{(n)}(a, z_0; z) \tag{37}$$

where the absence of the subindex m means that all the formal powers correspond to the same generating pair (F, G), and the series converges uniformly in some neighbourhood of z_0. It can be shown that the uniform limit of pseudoanalytic functions is pseudoanalytic and that a uniformly convergent series of (F, G)-pseudoanalytic functions can be (F, G) differentiated term by term. Hence, the function W in (37) is (F, G)-pseudoanalytic and its rth derivative admits the expansion

$$W^{[r]}(z) = \sum_{n=r}^{\infty} n(n-1)\cdots(n-r+1)Z_r^{(n-r)}(a_n, z_0; z).$$

From this the Taylor formulae for the coefficients are obtained

$$a_n = \frac{W^{[n]}(z_0)}{n!}. \tag{38}$$

Definition 22. *Let $W(z)$ be a given (F, G)-pseudoanalytic function defined for small values of $|z - z_0|$. The series*

$$\sum_{n=0}^{\infty} Z^{(n)}(a, z_0; z) \tag{39}$$

with the coefficients given by (38) is called the Taylor series of W at z_0, formed with formal powers.

The Taylor series always represents the function asymptotically:

$$W(z) - \sum_{n=0}^{N} Z^{(n)}(a, z_0; z) = O(|z - z_0|^{N+1}), \qquad z \to z_0, \tag{40}$$

for all N.

If the series (39) converges uniformly in a neighbourhood of z_0, it converges to the function W.

4.3. Convergence theorems

The statements given in this subsection were obtained by Bers [5], [7] and Agmon and Bers [1]. Their proof in a usual complex case was based on the so-called similarity principle. The similarity principle in general is not valid in a bicomplex situation. Here, we correct the corresponding statement which unfortunately in [9] was formulated with a mistake.

Theorem 23 (similarity principle)**.** *Let w be a regular solution of (33) in a domain Ω such that its values are not zero divisors at any point. Then the bicomplex function $\Phi = w \cdot e^h$, where*

$$h(z) = \frac{1}{\pi} \int_{\Omega} \frac{g(\tau)\,\mathrm{d}\tau}{\tau - z},$$

$$g(z) = \begin{cases} a_{(F,G)}(z) + b_{(F,G)}(z)\dfrac{\overline{w}(z)}{w(z)} & \text{if } w(z) \neq 0, \quad z \in \Omega \\ a_{(F,G)}(z) + b_{(F,G)}(z) & \text{if } w(z) = 0, \quad z \in \Omega \end{cases}$$

is a solution of the equation $\partial_{\bar{z}}\Phi = 0$ in Ω.

The proof is completely analogous to that for a complex case (see [26]).

In the case when the coefficients in (33) are usual complex functions (with respect to k) the following theorems regarding the convergence of formal Taylor expansions are valid.

Theorem 24 [5]**.** *The formal Taylor expansion (39) of a pseudoanalytic function in formal powers defined by a periodic generating sequence converges in some neighbourhood of the centre.*

Definition 25 [5]. *A generating pair (F, G) is called complete if these functions are defined and satisfy the Hölder condition for all finite values of z, the limits $F(\infty), G(\infty)$ exist,* $\mathrm{Vec}(\overline{F(\infty)}G(\infty)) > 0$ *and the functions $F(1/z), G(1/z)$ also satisfy the Hölder condition. A complete generating pair is called normalized if $F(\infty) = 1$, $G(\infty) = k$.*

A generating pair equivalent to a complete one is complete, and every complete generating pair is equivalent to a uniquely determined normalized pair. The adjoint of a complete (normalized) generating pair is complete (normalized).

From now on we assume that (F, G) is a complete normalized generating pair. Then much more can be said on the series of corresponding formal powers. We limit ourselves to the following completeness results (the expansion theorem and Runge's approximation theorem for pseudoanalytic functions).

Following [5] we shall say that a sequence of functions W_n converges normally in a domain Ω if it converges uniformly on every bounded closed subdomain of Ω.

Theorem 26. *Let W be an (F, G)-pseudoanalytic function defined for $|z - z_0| < R$. Then it admits a unique expansion of the form $W(z) = \sum_{n=0}^{\infty} Z^{(n)}(a_n, z_0; z)$ which converges normally for $|z - z_0| < \theta R$, where θ is a positive constant depending on the generating sequence.*

The first version of this theorem was proved in [1]. We follow here [7].

Remark 27. Necessary and sufficient conditions for the relation $\theta = 1$ are, unfortunately, not known. However, in [7] the following sufficient conditions for the case when the generators (F, G) possess partial derivatives are given. One such condition reads

$$|F_{\overline{z}}(z)| + |G_{\overline{z}}(z)| \leqslant \frac{\mathrm{const}}{1 + |z|^{1+\varepsilon}}$$

for some $\varepsilon > 0$. Another condition is

$$\iint_{|z|<\infty} (|F_{\overline{z}}|^{2-\varepsilon} + |F_{\overline{z}}|^{2+\varepsilon} + |G_{\overline{z}}|^{2-\varepsilon} + |G_{\overline{z}}|^{2+\varepsilon})\, \mathrm{d}x\, \mathrm{d}y < \infty$$

for some $0 < \varepsilon < 1$.

Theorem 28 [7]. *A pseudoanalytic function defined in a simply connected domain can be expanded into a normally convergent series of formal polynomials (linear combinations of formal powers with positive exponents).*

Remark 29. This theorem admits a direct generalization onto the case of a multiply connected domain (see [7]).

In posterior works [11, 14, 22], deep results on interpolation and on the degree of approximation by pseudopolynomials were obtained. For example,

Theorem 30 [22]. *Let W be a pseudoanalytic function in a domain Ω bounded by a Jordan curve and satisfy the Hölder condition on $\partial\Omega$ with the exponent α $(0 < \alpha \leqslant 1)$. Then for any $\varepsilon > 0$ and any natural n there exists a pseudopolynomial of order n satisfying the inequality*

$$|W(z) - P_n(z)| \leqslant \frac{\mathrm{const}}{n^{\alpha-\varepsilon}} \qquad \text{for any} \quad z \in \overline{\Omega}$$

where the constant does not depend on n, but only on ε.

The primary aim of the next two sections is to show that

(1) all the mentioned results are of immediate application to equation (13),
(2) in many practically important situations the generating sequence and consequently the formal powers $Z^{(n)}$, $n = 0, 1, \ldots,$ can be constructed explicitly.

5. Complete systems of solutions for second-order equations

In what follows let us suppose that the scalar function f is defined in a somewhat bigger domain Ω_ε with a sufficiently smooth boundary. Then we change the function f for $z \in \Omega_\varepsilon \backslash \Omega$ and continue it over the whole plane in such a way that $f \equiv 1$ for large $|z|$ (see [7]). In this way, the generating pair $(F, G) = (f, k/f)$ becomes complete and normalized.

Then the following statements are direct corollaries of relations established in section 3 between pseudoanalytic functions (solutions of (18)) and solutions of second-order elliptic equations and convergence theorems from the previous section.

Definition 31. *Let $u(z)$ be a given solution of equation (13) defined for small values of $|z - z_0|$ and let $W(z)$ be a solution of (18) constructed according to corollary 13 such that $\mathrm{Sc} W = p^{1/2} u$. The series*

$$p^{-1/2}(z) \sum_{n=0}^{\infty} \mathrm{Sc} Z^{(n)}(a_n, z_0; z)$$

with the coefficients given by (38) is called the Taylor series of u at z_0, formed with formal powers.

In the rest of this section, we assume that all the coefficients in second-order equations considered in section 3 are real-valued functions and the particular nonvanishing solution u_0 of (13) is real valued as well.

Theorem 32. *Let $u(z)$ be a solution of (13) defined for $|z - z_0| < R$. Then it admits a unique expansion of the form*

$$u(z) = p^{-1/2}(z) \sum_{n=0}^{\infty} \mathrm{Sc}\, Z^{(n)}(a_n, z_0; z)$$

which converges normally for $|z - z_0| < R$.

Proof. This is a direct consequence of theorem 26 and remark 27. Both necessary conditions in remark 27 are fulfilled for the generating pair (24). □

Theorem 33. *An arbitrary solution of (13) defined in a simply connected domain where there exists a nonvanishing particular solution u_0 can be expanded into a normally convergent series of formal polynomials multiplied by $p^{-1/2}$.*

Proof. This is a direct corollary of theorem 28. □

More precisely the last theorem has the following meaning. Due to property 2 of formal powers we have that $Z^{(n)}(a, z_0; z)$ for any Taylor coefficient a can be easily expressed through $Z^{(n)}(1, z_0; z)$ and $Z^{(n)}(k, z_0; z)$. Then due to theorem 28 any solution W of (18) can be expanded into a normally convergent series of linear combinations of $Z^{(n)}(1, z_0; z)$ and $Z^{(n)}(k, z_0; z)$. Consequently, any solution of (13) can be expanded into a normally convergent series of linear combinations of scalar parts of $Z^{(n)}(1, z_0; z)$ and $Z^{(n)}(k, z_0; z)$ multiplied by $p^{-1/2}$.

Obviously, for solutions of (13) the results on the interpolation and on the degree of approximation like, e.g., theorem 30, are also valid.

Let us stress that theorem 33 gives us the following result. The functions

$$\{p^{-1/2}(z) \mathrm{Sc}\, Z^{(n)}(1, z_0; z),\ p^{-1/2}(z) \mathrm{Sc}\, Z^{(n)}(k, z_0; z)\}_{n=0}^{\infty} \tag{41}$$

represent a complete system of solutions of (13) in the sense that any solution of (13) can be represented by a normally convergent series formed by functions (41) in any simply connected domain Ω where a positive solution of (13) exists. Moreover, as we show in the next section, in many practically interesting situations these functions can be constructed explicitly.

6. Explicit construction of positive formal powers

The book [5] (see also [10, supplement to chapter 4]) contains explicit formulae for the calculation of positive formal powers in the case when F and G have the form

$$F = \left(\frac{\gamma(x)}{\tau(y)}\right)^{1/2} \qquad \text{and} \qquad G = k\left(\frac{\gamma(x)}{\tau(y)}\right)^{-1/2}.$$

In [18] the class of generating pairs for which the generating sequence and hence the corresponding formal powers can be constructed explicitly was substantially extended. For the generating pair of the form (24) it is possible when f fulfils the following condition.

Condition 34 (condition S) [18]**.** *Let f be a scalar function of some real variable $\rho = \rho(x, y)$: $f = f(\rho)$ such that the expression $\frac{\Delta\rho}{|\nabla\rho|^2}$ is a function of ρ. We denote it by $s(\rho) = \frac{\Delta\rho}{|\nabla\rho|^2}$.*

Besides the obvious example of any harmonic function ρ and as a consequence of ρ being a Cartesian variable or $\rho = \arg z = \arctan(y/x)$, there are many other practically important examples of ρ satisfying condition S. An important example is $\rho(x, y) = \sqrt{x^2 + y^2}$. In this case $s(\rho) = \frac{1}{\rho}$. The parabolic coordinate $\rho(x, y) = \sqrt{x^2 + y^2} + x$ also fulfils condition S: $s(\rho) = \frac{1}{2\rho}$. Elliptic coordinates fulfil condition S as well (see [18]). In fact, as we will discuss in detail in a forthcoming publication, condition S can be interpreted as follows. Let $\xi = \xi(x, y)$ and $\eta = \eta(x, y)$ represent an arbitrary orthogonal coordinate system on the plane and ρ be an arbitrary (nonconstant) twice differentiable real-valued function of ξ or of η. Then $f = f(\rho)$ satisfies the condition S.

Denote by S an antiderivative of s with respect to ρ.

Theorem 35 [18]**.** *Let f be a scalar function of a real variable ρ satisfying condition S and let the function $\varphi = k\mathrm{e}^{-S}\rho_z$ have no zeros and be bounded in Ω. Then the generating pair (F, G) with $F = f$ and $G = k/f$ is embedded in the generating sequence (F_m, G_m), $m = 0, \pm1, \pm2, \ldots$, with $F_m = \varphi^m F$ and $G_m = \varphi^m G$.*

This result opens the way for explicit construction of positive formal powers for equation (18) and as a consequence of the complete system of solutions (41) for equation (13).

Some examples of explicitly constructed formal powers were given in [18]. Here, we show another quite simple but illustrative example.

Example 36. Consider the Helmholtz equation

$$(-\Delta + c^2)u = 0 \tag{42}$$

with c being a real constant. Take the following particular solution of (42): $f = \mathrm{e}^{cy}$. Let us construct the first few corresponding formal powers with centre at the origin. We have

$$Z^{(0)}(1, 0; z) = \mathrm{e}^{cy}, \qquad Z^{(0)}(k, 0; z) = k\,\mathrm{e}^{-cy},$$

$$Z^{(1)}(1, 0; z) = x\,\mathrm{e}^{cy} + \frac{k\sinh(cy)}{c}, \qquad Z^{(1)}(k, 0; z) = -\frac{\sinh(cy)}{c} + kx\,\mathrm{e}^{-cy},$$

$$Z^{(2)}(1,0;z) = \left(x^2 - \frac{y}{c}\right)\mathrm{e}^{cy} + \frac{\sinh(cy)}{c^2} + \frac{2kx\sinh(cy)}{c},$$
$$Z^{(2)}(k,0;z) = -\frac{2x\sinh(cy)}{c} + k\left(\left(x^2 + \frac{y}{c}\right)\mathrm{e}^{-cy} - \frac{\sinh(cy)}{c^2}\right), \ldots .$$

It is a simple exercise to verify that indeed the asymptotic formulae (36) hold. Now taking scalar parts of the formal powers we obtain a complete system of solutions of the Helmholtz equation:

$$u_1(x,y) = \mathrm{e}^{cy}, \qquad u_2(x,y) = x\,\mathrm{e}^{cy}, \qquad u_3(x,y) = -\frac{\sinh(cy)}{c},$$
$$u_4(x,y) = \left(x^2 - \frac{y}{c}\right)\mathrm{e}^{cy} + \frac{\sinh(cy)}{c^2}, \qquad u_5(x,y) = -\frac{2x\sinh(cy)}{c}, \ldots .$$

Formal powers of higher order can be constructed explicitly using a computer system of symbolic calculation. For this particular example (together with Maria Rosalía Tenorio) Matlab 6.5 allowed us to obtain analytic expressions for the formal powers up to the order 10 that gave us the first 21 functions $u_1, \ldots, u_{21}$. We used them for a numerical solution of the Dirichlet problem for the Helmholtz equation with very satisfactory results. For example, in the case when Ω is a unit disc with centre at the origin, $c = 1$ and u on the boundary is equal to e^x (this test exact solution gave us the worst precision because of its obvious 'disparateness' from functions $u_1, u_2 \ldots$) the maximal error $\max_{z\in\Omega} |u(z) - \widetilde{u}(z)|$ where u is the exact solution and $\widetilde{u} = \sum_{n=1}^{21} a_n u_n$, the real constants a_n being found by the collocation method, was of order 10^{-7}. A very fast convergence of the method was observed.

Although the numerical method based on the usage of explicitly or numerically constructed pseudoanalytic formal powers still needs a much more detailed analysis, these first results show us that it is quite possible that in due time and with a further development of symbolic calculation systems it can rank high among other numerical approaches, especially for solving equations (8) or (13) with rapidly varying coefficients, when finite-difference methods fail.

7. Reduction of the multidimensional second-order equation to a first-order equation

Here, we consider the case of dimension $n = 3$ and in the final part of this section we show that a simple generalization gives us the same results in higher dimensions.

We will consider the algebra $\mathbb{H}(\mathbb{C})$ of complex quaternions or biquaternions which have the form $Q = Q_0 + Q_1\mathbf{i} + Q_2\mathbf{j} + Q_3\mathbf{k}$, where $\{Q_k\} \subset \mathbb{C}$, and $\mathbf{i}, \mathbf{j}, \mathbf{k}$ are the quaternionic imaginary units.

The vectorial representation of a complex quaternion will be used. Namely, each complex quaternion Q is a sum of a scalar Q_0 and of a vector $\boldsymbol{Q}$:

$$Q = \mathrm{Sc}(Q) + \mathrm{Vec}(Q) = Q_0 + \mathbf{Q},$$

where $\mathbf{Q} = Q_1\mathbf{i} + Q_2\mathbf{j} + Q_3\mathbf{k}$. The operator of quaternionic conjugation we denote by C_H: $\overline{Q} = C_H Q = Q_0 - \mathbf{Q}$. We conserve the bar for the quaternionic conjugation which should not provoke any confusion with the same notation for the conjugation in the first part of the paper because essentially it can be considered as the same operation if the bicomplex numbers are considered being embedded in $\mathbb{H}(\mathbb{C})$ in a natural way.

The purely vectorial complex quaternions ($\mathrm{Sc}(Q) = 0$) are identified with vectors from $\mathbb{C}^3$. Note that $\mathbf{Q}^2 = -\langle \mathbf{Q}, \mathbf{Q}\rangle$ where $\langle\cdot,\cdot\rangle$ denotes the usual scalar product.

By M^P we denote the operator of multiplication by a complex quaternion P from the right-hand side: $M^P Q = Q \cdot P$. More information on the structure of the algebra of complex quaternions can be found, for example, in [16] or [21].

Let Q be a complex quaternion-valued differentiable function of $\mathbf{x} = (x, y, z)$. Denote

$$DQ = \mathbf{i}\frac{\partial}{\partial x}Q + \mathbf{j}\frac{\partial}{\partial y}Q + \mathbf{k}\frac{\partial}{\partial z}Q.$$

This expression can be rewritten in a vector form as follows:

$$DQ = -\mathrm{div}\,\mathbf{Q} + \mathrm{grad}\,Q_0 + \mathrm{rot}\,\mathbf{Q}.$$

That is, $\mathrm{Sc}\,(DQ) = -\mathrm{div}\,\mathbf{Q}$ and $\mathrm{Vec}(DQ) = \mathrm{grad}\,Q_0 + \mathrm{rot}\,\mathbf{Q}$. Let us note that $D^2 = -\Delta$. If Q_0 is a scalar function then DQ_0 coincides with $\mathrm{grad}\,Q_0$.

The following generalization of Leibniz's rule can be proved by a direct calculation (see [12, p 24]).

Theorem 37 (generalized Leibniz rule)**.** *Let* $\{P, Q\} \subset C^1(G; \mathbb{H}(\mathbb{C}))$*, where* G *is some domain in* $\mathbb{R}^3$*. Then*

$$D[P\cdot Q] = D[P]\cdot Q + \overline{P}\cdot D[Q] + 2(\mathrm{Sc}(PD))[Q], \tag{43}$$

where

$$(\mathrm{Sc}(PD))[Q] := -\sum_{j=1}^{3} P_j \partial_j Q.$$

We will actively use the following:

Remark 38. If in theorem 37 $\mathrm{Vec}(P) = 0$, that is $P = P_0$, then

$$D[P_0\cdot Q] = D[P_0]\cdot Q + P_0\cdot D[Q]. \tag{44}$$

From this equality we obtain that the operator $D + \frac{\mathrm{grad}\,P_0}{P_0}$ can be factorized as follows:

$$\left(D + \frac{\mathrm{grad}\,P_0}{P_0}\right)Q = P_0^{-1}D(P_0\cdot Q). \tag{45}$$

Let $\mathbf{G}$ be a complex-valued vector such that $\mathrm{rot}\,\mathbf{G} \equiv 0$. Then the complex-valued scalar function φ is said to be its antigradient if $\mathrm{grad}\,\varphi = \mathbf{G}$. We will write $\varphi = \mathcal{A}[\mathbf{G}]$. The operator $\mathcal{A}$ is a simple generalization of the usual antiderivative and of the operator $\overline{A}$ (see section 2), and it defines the function φ up to an arbitrary constant. Its explicit representation is well known and has the form

$$\mathcal{A}[\mathbf{G}](x, y, z) = \int_{x_0}^{x} G_1(\xi, y_0, z_0)\,\mathrm{d}\xi + \int_{y_0}^{y} G_2(x, \zeta, z_0)\,\mathrm{d}\zeta + \int_{z_0}^{z} G_3(x, y, \eta)\,\mathrm{d}\eta + C.$$

Consider

$$(-\Delta + \nu)g = 0 \qquad \text{in} \quad G \tag{46}$$

where $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$, ν and g are the complex-valued functions, and G is a domain in $\mathbb{R}^3$. We assume that g is twice continuously differentiable.

Theorem 39. *Let* f *be a nonvanishing particular solution of (46). Then for any scalar twice continuously differentiable function* g *the following equality holds:*

$$\left(D + M^{\frac{Df}{f}}\right)\left(D - M^{\frac{Df}{f}}\right)g = (-\Delta + \nu)g. \tag{47}$$

Proof. This is a direct calculation based on the Leibniz rule (44). □

Remark 40. The factorization (47) was obtained in [3, 4] in a form which required a solution of an associated biquaternionic Riccati equation. In [15] it was shown that the solution has necessarily the form Df/f with f being a solution of (46).

Remark 41. Theorem 39 generalizes theorem 1. In a two-dimensional situation (47) reduces to (9).

Remark 42. As g in (47) is a scalar function, the factorization of the Schrödinger operator can be written in the following form:

$$\left(D + M^{\frac{Df}{f}}\right) f D(f^{-1}g) = (-\Delta + \nu)g,$$

from which it is obvious that if g is a solution of (46) then the vector $\mathbf{F} = fD(f^{-1}g)$ is a solution of

$$\left(D + M^{\frac{Df}{f}}\right)\mathbf{F} = 0 \qquad \text{in} \quad G. \tag{48}$$

The inverse result we formulate as the following statement.

Theorem 43. *Let* $\mathbf{F}$ *be a solution of (48) in a simply connected domain* G*. Then* $g = f\mathcal{A}[f^{-1}\mathbf{F}]$ *is a solution of (46).*

Proof. First, in order to apply the operator $\mathcal{A}$ to the vector $f^{-1}\mathbf{F}$ we should ascertain that indeed

$$\operatorname{rot}(f^{-1}\mathbf{F}) = 0. \tag{49}$$

For this, consider the vector part of (48). It has the form

$$\operatorname{rot}\mathbf{F} + \left[\mathbf{F} \times \frac{Df}{f}\right] = 0$$

which is equivalent to equation (49).

Now, applying the Laplacian to $g = f\mathcal{A}[f^{-1}\mathbf{F}]$ and taking into account that f is a solution of (46) and $\mathbf{F}$ is a solution of (48) we obtain the result

$$\begin{aligned} -\Delta g &= D^2 g = D(Df \cdot \mathcal{A}[f^{-1}\mathbf{F}] + \mathbf{F}) \\ &= f^{-1}\mathbf{F}Df - \mathcal{A}[f^{-1}\mathbf{F}]\Delta f + D\mathbf{F} \\ &= \mathbf{F}\frac{Df}{f} - \nu f\mathcal{A}[f^{-1}\mathbf{F}] - \mathbf{F}\frac{Df}{f} \\ &= -\nu g. \end{aligned}$$

□

In the same way as in section 3 we obtain the factorization of the operator $\operatorname{div} p \operatorname{grad} + q$ where div and grad are already operators with respect to three independent variables.

Theorem 44. *Let* u_0 *be a nonvanishing particular solution of*

$$(\operatorname{div} p \operatorname{grad} + q)u = 0 \qquad \textit{in} \quad G \subset \mathbb{R}^3 \tag{50}$$

with p, q *and* u *being complex-valued functions,* $p \in C^2(G)$ *and* $p \neq 0$ *in* G*. Then for any scalar function* $\varphi \in C^2(G)$ *the following equality holds:*

$$(\operatorname{div} p \operatorname{grad} + q)\varphi = -p^{1/2}\left(D + M^{\frac{Df}{f}}\right)\left(D - M^{\frac{Df}{f}}\right)p^{1/2}\varphi \tag{51}$$

where $f = p^{1/2}u_0$*.*

Proof. This is analogous to the proof of theorem 3. □

Thus, if u is a solution of equation (50) then

$$\mathbf{F} = fD(f^{-1}p^{1/2}u) = fD\big(u_0^{-1}u\big)$$

is a solution of equation (48) (see remark 42). The inverse result has the following form.

Theorem 45. *Let* $\mathbf{F}$ *be a solution of equation (48) in a simply connected domain* G, *where* $f = p^{1/2}u_0$ *and* u_0 *be a nonvanishing particular solution of (50). Then*

$$u = u_0\mathcal{A}[f^{-1}\mathbf{F}]$$

is a solution of (50).

Proof. This is a corollary of theorem 43 and relation $(\operatorname{div} p \operatorname{grad} + q) = p^{1/2}(\Delta - \nu)p^{1/2}$ where $\nu = \Delta f/f$. □

Note that due to the fact that in (51) φ is scalar, we can rewrite the equality in the form

$$(\operatorname{div} p \operatorname{grad} + q)\varphi = -p^{1/2}\big(D + M^{\frac{Df}{f}}\big)\left(D - \frac{Df}{f}C_H\right)p^{1/2}\varphi.$$

Now, consider

$$\left(D - \frac{Df}{f}C_H\right)W = 0, \tag{52}$$

where W is an $\mathbb{H}(\mathbb{C})$-valued function. Equation (52) is a direct generalization of the Vekua equation (18). Moreover, we show that it preserves some important properties of (18).

Theorem 46. *Let* $W = W_0 + \mathbf{W}$ *be a solution of (52). Then* W_0 *is a solution of the stationary Schrödinger equation*

$$-\Delta W_0 + \nu W_0 = 0, \tag{53}$$

where $\nu = \Delta f/f$. *Moreover, the function* $u = f^{-1}W_0$ *is a solution of*

$$\operatorname{div}(f^2 \operatorname{grad} u) = 0 \tag{54}$$

and the vector function $\mathbf{v} = f\mathbf{W}$ *is a solution of*

$$\operatorname{rot}(f^{-2} \operatorname{rot} \mathbf{v}) = 0. \tag{55}$$

Proof. Equation (52) is equivalent to the system

$$\operatorname{div} \mathbf{W} + \left\langle \frac{\nabla f}{f}, \mathbf{W}\right\rangle = 0,$$
$$\operatorname{rot} \mathbf{W} + \left[\frac{\nabla f}{f} \times \mathbf{W}\right] + \nabla W_0 - \frac{\nabla f}{f}W_0 = 0$$

which can be rewritten in the form

$$\operatorname{div}(f\mathbf{W}) = 0, \tag{56}$$

$$f^{-1}\operatorname{rot}(f\mathbf{W}) + f\operatorname{grad}(f^{-1}W_0) = 0. \tag{57}$$

From (57) we obtain (54) and (55). Equation (53) is obtained from (54) and (12). □

Remark 47. Observe that the functions

$$F_0 = f, \qquad F_1 = \frac{\mathbf{i}}{f}, \qquad F_2 = \frac{\mathbf{j}}{f}, \qquad F_3 = \frac{\mathbf{k}}{f}$$

give us a generating quartet for equation (52). They are solutions of (52) and obviously any $\mathbb{H}(\mathbb{C})$-valued function W can be represented in the form

$$W = \sum_{j=0}^{3} \varphi_j F_j,$$

where φ_j are complex-valued functions. It is easy to verify that the function W is a solution of (52) iff

$$\sum_{j=0}^{3} (D\varphi_j) F_j = 0 \tag{58}$$

in a complete analogy with the two-dimensional case (see remark 8). Denote

$$w = \varphi_0 + \varphi_1 \mathbf{i} + \varphi_2 \mathbf{j} + \varphi_3 \mathbf{k}.$$

Then (58) can be written as follows:

$$D(w + \overline{w}) f + D(w - \overline{w}) \frac{1}{f} = 0$$

which is equivalent to

$$Dw = \frac{1 - f^2}{1 + f^2} D\overline{w}.$$

Remark 48. The results of this section remain valid in the n-dimensional situation if instead of quaternions the Clifford algebra $Cl_{0,n}$ (see, e.g., [8], [13]) is considered. The operator D is then introduced as follows $D = \sum_{j=1}^{n} e_j \frac{\partial}{\partial x_j}$ where e_j are the basic basis elements of the Clifford algebra.

8. Conclusions

We showed the possibility of a factorization of the operator $(\operatorname{div} p \operatorname{grad} + q)$ and investigated only some of its applications. In a two-dimensional situation under quite general assumptions a complete system of null solutions of the operator can be constructed explicitly. It is quite possible that in a multidimensional case using the results of the preceding section the same can be done. This requires a multidimensional generalization of Bers' theory of formal powers.

Another open question is the proof of expansion and convergence theorems for the bicomplex Vekua equation of the form (18).

References

[1] Agmon S and Bers L 1952 The expansion theorem for pseudo-analytic functions *Proc. Am. Math. Soc.* **3** 757–64
[2] Astala K and Päivärinta L 2006 Calderón's inverse conductivity problem in the plane *Ann. Math.* **163** 265–99
[3] Bernstein S 1996 Factorization of solutions of the Schrödinger equation *Proc. Symposium Analytical and Numerical Methods in Quaternionic and Clifford Analysis (Seiffen)*
[4] Bernstein S and Gürlebeck K 1999 On a higher dimensional Miura transform *Complex Variables* **38** 307–19
[5] Bers L 1952 *Theory of Pseudo-Analytic Functions* (New York: New York University)
[6] Bers L 1956 An outline of the theory of pseudoanalytic functions *Bull. Am. Math. Soc.* **62** 291–331
[7] Bers L 1956 Formal powers and power series *Commun. Pure Appl. Math.* **9** 693–711
[8] Brackx F, Delanghe R and Sommen F 1982 *Clifford Analysis* (Marshfield, MA: Pitman)
[9] Castañeda A and Kravchenko V V 2005 New applications of pseudoanalytic function theory to the Dirac equation *J. Phys. A: Math. Gen.* **38** 9207–19
[10] Courant R and Hilbert D 1989 *Methods of Mathematical Physics* vol 2 (New York: Wiley-Interscience)

[11] Fryant A 1981 Ultraspherical expansions and pseudo analytic functions *Pac. J. Math.* **94** 83–105
[12] Gürlebeck K and Sprößig W 1989 *Quaternionic Analysis and Elliptic Boundary Value Problems* (Berlin: Akademie)
[13] Gürlebeck K and Sprössig W 1997 *Quaternionic and Clifford Calculus for Physicists and Engineers* (New York: Wiley)
[14] Ismailov A Ja and Tagieva M A 1970 On the representation of generalized analytic functions by a series of pseudopolynomials *Sov. Math. Dokl.* **11** 1605–8
[15] Kravchenko V G, Kravchenko V V and Williams B D 2001 A quaternionic generalization of the Riccati differential equation *Clifford Analysis and Its Applications* ed F Brackx *et al* (Dordrecht: Kluwer) pp 143–54
[16] Kravchenko V V 2003 *Applied Quaternionic Analysis* (Lemgo: Heldermann Verlag)
[17] Kravchenko V V 2005 On the reduction of the multidimensional stationary Schrödinger equation to a first order equation and its relation to the pseudoanalytic function theory *J. Phys. A: Math. Gen.* **38** 851–68
[18] Kravchenko V V 2005 On a relation of pseudoanalytic function theory to the two-dimensional stationary Schrödinger equation and Taylor series in formal powers for its solutions *J. Phys. A: Math. Gen.* **38** 3947–64
[19] Kravchenko V V 2005 On the relationship between p-analytic functions and the Schrödinger equation *Z. Anal. Anwendungen* **24** 487–96
[20] Kravchenko V V and Oviedo H On explicitly solvable Vekua equations and explicit solution of the stationary Schrödinger equation and of the equation $\operatorname{div}(\sigma\nabla u) = 0$ *Complex Variables and Elliptic Equations* (to appear)
[21] Kravchenko V V and Shapiro M V 1996 *Integral Representations for Spatial Models of Mathematical Physics* (*Pitman Res. Notes in Math. Series* vol 351) (Harlow: Addison-Wesley/Longman)
[22] Menke K 1974 Zur Approximation pseudoanalytischer Funktionen durch Pseudopolynome *Manuscr. Math.* **11** 111–25
[23] Nachman A 1988 Reconstructions from boundary measurements, *Ann. Math.* **128** 531–76
[24] Rochon D and Tremblay S 2004 Bicomplex quantum mechanics: I. The generalized Schrödinger equation *Adv. Appl. Clifford Algebr.* **14** 231–48
[25] Uhlmann G 1999 Developments in inverse problems since Calderón's foundational paper *Harmonic Analysis and Partial Differential Equations, Essays in Honor of Alberto P. Calderón* (*Chicago Lectures in Mathematics*) ed M Christ, C E Kenig and C Sadosky pp 295–345
[26] Vekua I N 1959 *Generalized Analytic Functions* (Moscow: Nauka) (in Russian)
Vekua I N 1962 *Generalized Analytic Functions* (Oxford: Pergamon) (Engl. Transl.)